1 Introduction

What follows is a treatment of the one-dimensional random walk that we have discussed in class. It is hoped that these notes will augment the work we have done during lecture.

2 Statement of the Problem

In the one-dimensional random walk, we imagine a one-dimensional creature, labeled by an “X” that begins to walk at the origin of a Cartesian coordinate system as pictured below:

The walker can take steps of unit length either to the left or to the right. At each step, the direction of the walk is chosen completely at random. For example, after one step the location of the walker will be either -1

or at +1
If the probabilities of left and right steps are equal, after one step each of the two possible locations are equally probable. The statement of equal probability means that if many walkers at the origin of their own coordinate systems make one random step each, 50 per cent of the walkers will find themselves at location +1 and 50 per cent will find themselves at location -1. After two steps the possible locations are +2

\[\begin{array}{cccccccc}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
\end{array}\]

or 0

\[\begin{array}{cccccccc}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
\end{array}\]

or -2

Note that two walkers have been placed at location 0. The two walkers at 0 represent the two ways the walkers can reach 0; a step to the left followed by a step to the right or a step to the right followed by a step to the left. In contrast, the walker can reach location -2 only by two steps to the left and +2 only by making two steps to the right. Consequently, if many walkers make two steps each, the fraction that will be at location 0 will be 1/2. Additionally, 1/4 of the walkers will be at location +2 and 1/4 of the walkers will be at location -2.

The probability distribution obtained by continuing the walk for more steps is given by a technique developed by Pascal. Known as Pascal’s triangle, it is a set of integers obtained by beginning with the number 1 in the first line and generating subsequent lines by addition of the integers in the previous line.
The form of Pascal’s triangle is

1 1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The meaning of the last line shown for Pascal’s triangle is that if a random walk is executed with a total of 6 steps, there are 20 ways of reaching the origin, 15 ways each of reaching locations ± 2, 6 ways each of reaching locations ± 4 and only 1 way each of reaching locations ± 6. If the number of ways is plotted in a histogram, the random walk distribution is qualitatively seen to approach that of a bell shaped curve. We make this more explicit in the next section.

3 The Random Walk Probability Distribution

We now need to make the discussion of the random walk quantitative. We first generalize the walk so that the probabilities of left and right steps are not necessarily equal. We let $p$ be the probability that the walker makes a step to the right (the positive direction) and $q$ be the probability that the walker makes a step to the left (which we take to be the negative direction). Clearly,

$$p + q = 1$$ (1)

We let the walker make $N$ steps, $n_R$ of which are to the right and $n_L$ of which are to the left. Then

$$n_R + n_L = N$$ (2)

If $m$ is the final location of the walker after $N$ steps, we also have the relation

$$n_R - n_L = m$$ (3)

We wish to develop an expression for the probability that the walker finds itself at location $m$ after $N$ steps. Notice that $m$, $N$, $n_R$ and $n_L$ are not all independent by virtue of Eqs. (2) and (3). In particular we can write

$$m = 2n_R - N$$ (4)

so that we can express the probability we seek either in terms of $m$ or $n_R$. We shall develop expressions for both. We first note that the probability of any particular sequence of left and right steps is given by $p^{n_R}q^{n_L}$. In giving this expression, we have assumed that the
probability of any one particular sequence is equal to the probability of any other particular sequence. Then the probability of the walker making \( n_R \) steps to the right and \( n_L \) steps to the left is given by the probability of each possible sequence multiplied by the number of possible sequences. By the binomial distribution, this probability is then given by

\[
P_N(n_R, n_L) = \frac{N!}{n_R!n_L!} p^{n_R} q^{n_L} \quad (5)
\]

Expressing Eq.(5) in terms of \( m \) alone or \( n_R \) alone we have

\[
P_N(n_R) = \frac{N!}{n_R!(N-n_R)!} p^{n_R} q^{N-n_R} \quad (6)
\]

or

\[
P_N(m) = \frac{N!}{[1/2(N+m)]![1/2(N-m)]!} p^{1/2(N+m)} q^{1/2(N-m)} \quad (7)
\]

4 The Width of the Random Walk Probability Distribution

An important property of the random walk probability distribution is the behavior of its width with respect to \( N \). The width of a distribution is measured by its standard deviation. It is first necessary to define averages with respect to a given probability distribution \( P(u) \). If \( u \) can take on \( N \) discrete values \( u_1, u_2, \ldots, u_N \), then the mean value of any function of \( u \), \( f(u) \) is defined by the equation

\[
\langle f(u) \rangle = \frac{\sum_{j=1}^{N} f(u_j) P(u_j)}{\sum_{j=1}^{N} P(u_j)} \quad (8)
\]

Often, the probability function \( P(u) \) is defined so that it satisfies a normalization condition; i.e.

\[
\sum_{j=1}^{N} P(u_j) = 1 \quad (9)
\]

so that Eq. (8) becomes

\[
\langle f(u) \rangle = \sum_{j=1}^{N} f(u_j) P(u_j) \quad (10)
\]

For two functions of \( u \), \( f(u) \) and \( g(u) \) and for a numerical constant \( c \), it is easy to prove the relation

\[
\langle f(u) + cg(u) \rangle = \langle f(u) \rangle + c \langle g(u) \rangle \quad (11)
\]

A particularly important quantity is the deviation of \( u \) from its mean, defined by

\[
\Delta u = u - \langle u \rangle \quad (12)
\]
In terms of $\Delta u$, we can define the $n'th$ moment of $u$ about the mean by $< (\Delta u)^n >$. For the particular case that $n = 2$ we have

$$< (\Delta u)^2 >= < (u - < u >)^2 >$$

$$= < u^2 > - < u >^2$$

This second moment of $u$ about its mean is often called the square of the **standard deviation** of the probability distribution. For a given probability distribution, the standard deviation is a measure of the width of the distribution. The measure of the width in terms of the standard deviation will be clarified in Section 6.

Returning to the problem of the one-dimensional random walk, we now calculate an expression for its standard deviation. We first show that the probability distribution of Eq. (6) is normalized to unity. To show the normalization we begin with an important relation known as the **binomial theorem**. For any two numerical constants $a$ and $b$ the binomial theorem takes the form

$$(a + b)^N = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} a^n b^{N-n}$$

Using the binomial theorem and Eq. (6) we obtain

$$\sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} p^{n_R} q^{N-n_R}$$

$$= (p + q)^N$$

$$= 1$$

by virtue of Eq. (1). Because Eq. (6) has now been shown to be a normalized probability distribution, we can write

$$< n_R >= \sum_{n_R=0}^{N} n_R \frac{N!}{n_R!(N-n_R)!} p^{n_R} q^{N-n_R}$$

$$= \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} n_R p^{n_R}$$

Using the identity

$$np^n = \frac{d}{dp} p^n$$

Eq. (20) becomes

$$< n_R >= \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} p^{n_R}$$

$$= p \frac{d}{dp} \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} p^{n_R}$$

$$= p \frac{d}{dp} \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} p^{n_R}$$
\[ p \frac{\partial}{\partial p} (p + q)^N = Np \]  \hspace{1cm} \text{(24)}

an intuitively appealing result. Using similar methods, it is easy to show that \[ < n_L > = Nq. \]

The demonstration of the latter result is left as an exercise. To obtain the standard deviation of the random walk probability distribution, we need an expression for \[ < n_R^2 > \] which is now obtained using the same approach. We have

\[ < n_R^2 > = \sum_{n_R=0}^{N} n_R^2 \frac{N!}{n_R!(N-n_R)!} p^{n_R} q^{N-n_R} \]  \hspace{1cm} \text{(26)}

\[ = \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} n_R^2 p^{n_R} \]  \hspace{1cm} \text{(27)}

Using the identity

\[ n^2 p^n = p \frac{\partial}{\partial p} p^n + p^2 \frac{\partial^2}{\partial p^2} p^n \]  \hspace{1cm} \text{(28)}

we obtain

\[ < n_R^2 > = \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} q^{N-n_R} (p \frac{\partial}{\partial p} p^{n_R} + p^2 \frac{\partial^2}{\partial p^2} p^{n_R}) \]  \hspace{1cm} \text{(29)}

\[ = (p \frac{\partial}{\partial p} + p^2 \frac{\partial^2}{\partial p^2}) \sum_{n_R=0}^{N} \frac{N!}{n_R!(N-n_R)!} p^{n_R} q^{N-n_R} \]  \hspace{1cm} \text{(30)}

\[ = (p \frac{\partial}{\partial p} + p^2 \frac{\partial^2}{\partial p^2})(p + q)^N \]  \hspace{1cm} \text{(31)}

\[ = N^2 p^2 - Np^2 + Np \]  \hspace{1cm} \text{(32)}

Then

\[ \sigma^2 = < n_R^2 > - < n_R >^2 \]  \hspace{1cm} \text{(33)}

\[ = N^2 p^2 - Np^2 + Np - (Np)^2 \]  \hspace{1cm} \text{(34)}

\[ = Np(1 - p) = Npq \]  \hspace{1cm} \text{(35)}

Equation (35) implies that the standard deviation of the random walk probability distribution function grows as the square root of the number of steps. A measure of the relative width of the distribution is given by

\[ \sigma_N = \frac{\sigma}{< n_R >} \]  \hspace{1cm} \text{(36)}

\[ = \sqrt{\frac{q/p}{1/N}} \]  \hspace{1cm} \text{(37)}

The relative width decreases as the inverse of the square root of the number of steps in the walk. The decrease in the relative width with the square root of \( N \) is a very important property of the distribution.
5 Behavior for large $N$

We now develop an expression for the random walk probability distribution in the limit as the number of steps becomes large. We will begin with Eq. (7) and be interested in the limit that $N \to \infty$ under conditions where $m << N$. For simplicity we will specialize to the case that $p = q = 1/2$. The extension of the result to the more general case will be given in the next section.

The development requires the introduction of Stirling’s approximation which is exact in the limit of infinite $N$

$$
\ln N! \cong (N + \frac{1}{2}) \ln N - N + \frac{1}{2} \ln(2\pi) \quad (38)
$$

Using Stirling’s approximation and Eq. (7) we can write

$$
\lim_{N \to \infty} \ln P_N(m) = (N + \frac{1}{2}) \ln N - N + \frac{1}{2} \ln(2\pi) - \frac{1}{2} (N + m + 1) \ln \left[\frac{1}{2} (N + m)\right] + \frac{1}{2} (N - m) \ln \left[\frac{1}{2} (N - m)\right] - \frac{1}{2} (N + m + 1) \ln \left[\frac{1}{2} (N + m)\right] - \frac{1}{2} (N - m + 1) \ln \left[\frac{1}{2} (N - m)\right] \quad (39)
$$

$$
= N \ln N + \frac{1}{2} \ln(2\pi) - N \ln 2 - \frac{1}{2} (N + m + 1) \ln \left[\frac{1}{2} (N + m)\right] - \frac{1}{2} (N - m + 1) \ln \left[\frac{1}{2} (N - m)\right] \quad (40)
$$

$$
= N \ln N + \frac{1}{2} \ln(2\pi) - N \ln 2 - \frac{1}{2} (N + m + 1) \ln \left[\frac{N}{2} \left(1 + \frac{m}{N}\right)\right] - \frac{1}{2} (N - m + 1) \ln \left[\frac{N}{2} \left(1 - \frac{m}{N}\right)\right] \quad (41)
$$

$$
= N \ln N + \frac{1}{2} \ln(2\pi) - N \ln 2 - \frac{1}{2} (N + m + 1) \ln \left[\frac{N}{2} \right] - \frac{1}{2} (N - m + 1) \ln \left[\frac{N}{2} \right] \quad (42)
$$

We now introduce the Taylor expansion for the natural logarithm

$$
\ln(1 \pm x) = \pm x - 1/2x^2 + \ldots \quad (43)
$$

and keep terms to order $N^{-1}$ to obtain

$$
\lim_{N \to \infty} \ln P_N(m) = N \ln N + \frac{1}{2} \ln(2\pi) - N \ln 2 - N \ln \frac{N}{2} - N \ln \frac{N}{2} - \frac{1}{2} (N + m + 1) \ln \left[\frac{m}{N}\right] - \frac{1}{2} (N - m + 1) \ln \left[\frac{m}{N}\right] \quad (44)
$$

$$
= -\frac{1}{2} \ln N - \frac{1}{2} \ln(2\pi) + \ln 2 - \frac{m^2}{N} \quad (45)
$$

$$
= \ln \frac{2}{\sqrt{2\pi N}} - \frac{m^2}{2N} \quad (46)
$$
Taking the exponential of each side of Eq. (46), we finally obtain

$$\lim_{N \to \infty} P_N(m) = \left(\frac{2}{\pi N}\right)^{1/2} e^{-m^2/2N}$$  \hspace{1cm} (47)

Equation (47) is in the form of a continuous probability distribution in \(m\). It is usual to express the probability in terms of a continuous displacement coordinate \(x\). If we imagine the one-dimensional walker taking steps of length \(\ell\), then \(x = m\ell\), and Eq. (47) takes the form

$$P_N(x) = A e^{-x^2/2\sigma^2}$$ \hspace{1cm} (48)

In Eq. (48) \(A\) is a normalization factor to be discussed in the next section, and

$$\sigma = \sqrt{N\ell^2} \hspace{1cm} (49)$$

Equation (48) is called a Gaussian probability distribution. The Gaussian probability distribution is so important to statistical mechanics, that we discuss its properties in a separate section.

### 6 Properties of the Gaussian Distribution

Suppose \(P(x)\) is a general probability distribution of a continuous variable \(x\). We interpret \(P(x)dx\) to be the probability that an event occurs between \(x\) and \(x + dx\). Notice that the probability at one point \(x\) is not well defined, because there are an infinite number of outcomes possible when \(x\) is a continuous variable. We also interpret \(\int_a^b P(x)dx\) to be the probability that the event occurs between \(x = a\) and \(x = b\). For continuous probability distributions, the normalization condition takes the form

$$\int_D P(x)dx = 1$$ \hspace{1cm} (50)

where \(D\) represents the domain of the problem. We now wish to normalize the Gaussian distribution obtained from the large \(N\) limit of the random walk problem. We first need the result of an important general integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$ \hspace{1cm} (51)

Then from Eq. (48)

$$\int_{-\infty}^{\infty} A e^{-x^2/2\sigma^2} dx = A \sqrt{2\pi\sigma^2} \hspace{1cm} (52)$$

so that the normalized Gaussian distribution is given by

$$P_N(x) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp(-x^2/2\sigma^2)$$ \hspace{1cm} (53)
Before continuing, we need to generalize the random walk distribution for large \( N \) for the case where \( p \neq q \). In that case the normalized Gaussian distribution is of the form

\[
P_N(x) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\] (54)

where \( \mu \) is a parameter, the meaning of which will be made clear shortly. For continuous probability distribution, averages are defined similarly to the discrete case. If \( f(x) \) is some general function of \( x \), we define the average or mean of \( f \) by the relation

\[
\langle f(x) \rangle = \frac{\int_{-\infty}^{\infty} dx P(x) f(x)}{\int_{-\infty}^{\infty} dx P(x)}
\] (55)

where the denominator of Eq.(55) can be ignored if \( P(x) \) is normalized.

We now give expressions for the mean and standard deviation from the mean of \( x \) for a Gaussian probability distribution. The proof of these expressions is left as an exercise.

\[
\langle x \rangle = \mu
\] (56)

\[
\langle x^2 \rangle - \langle x \rangle^2 = \sigma^2
\] (57)

We see then that for a Gaussian distribution, the parameter \( \mu \) represents the mean, and the parameter \( \sigma^2 = N\ell^2 \) represents the square of the standard deviation. We can understand the meaning of \( \sigma \) better by examining the graph of the Gaussian distribution shown as the light line in Figure 1. For the light line of Figure 1 we have set \( \sigma = .1 \), and we have retained the overall normalization to unity. We have also put \( \mu = 0 \), arbitrarily. Notice that the peak of the distribution is centered about \( x = \mu \) as expected. In the dark solid line of Figure 1, we
also plot the normalized Gaussian distribution peaked about \( x = \mu = 0 \), but in this case we have taken \( \sigma = .05 \). Notice that the effect of decreasing \( \sigma \) is to make the overall distribution more narrow while increasing the peak height to maintain the normalization. This is one indication that \( \sigma \) is a measure of the width of the Gaussian distribution.

Another way of understanding \( \sigma \) as a width is by integrating the distribution from \(-\sigma\) to \(\sigma\) (notice \( \sigma \) has units of length). This integration must be performed numerically, because the indefinite integral of a Gaussian is not analytic \(^1\). By using techniques of numerical integration, one can show that

\[
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\sigma}^{\sigma} dx e^{-x^2/2\sigma^2} \approx .67
\]

i.e., approximately two thirds of the area under a Gaussian distribution lies between \( x = -\sigma \) and \( x = \sigma \). As \( \sigma \) becomes smaller, the distribution must become sharper so that most of the area below it will lie in the region within its standard deviation.

A third way of understanding the relation between the standard deviation and the width of a distribution is by defining the ratio of the height of the distribution at \( x = \sigma \) to its maximum at \( x = 0 \); i.e. we define \( R \) by

\[
R = \frac{e^{-\sigma^2/2\sigma^2}}{e^{0}} = e^{-0.5}
\]

We see that at \( x = \sigma \) the Gaussian has fallen to \( 1/\sqrt{\pi} \) of its maximum value. As \( \sigma \) becomes smaller, the distribution approaches zero more and more rapidly with \( x \).

7 The Central Limit Theorem

We close these notes with a very brief discussion of the **Central Limit Theorem**. The Central Limit Theorem is, perhaps, the most important in probability theory. It is a statement of the great generality of the Gaussian distribution. We have proved that the Gaussian distribution results from the one-dimensional random walk in the limit of many steps. According to the Central Limit Theorem a Gaussian distribution will arise from a random walk executed with step sizes chosen from any arbitrary but normalizable probability distribution (there are some additional limitations about how rapidly the distribution must approach zero at its extremes). Since many processes can be mapped onto such random walk distributions, a large class of statistically independent phenomenology will fall on a Gaussian distribution. For example, the process of flipping a fair coin can be shown to be identical to the one-dimensional random walk with fixed step sizes already considered here. Consequently, the probability distribution for flipping a coin approaches a Gaussian in the limit of many flips.

\(^1\)The integral over a Gaussian from 0 to any point \( x \) is defined to be the *error function* and is often tabulated.
Another example is experimental measurements in a laboratory. If a set of measurements of some quantity is made in a laboratory, the results will be distributed about the mean in a Gaussian distribution if the measurements are strictly statistically independent. As we shall see later in the semester the energies of member subsystems in a canonical ensemble fall on a Gaussian distribution about the mean energy of the system. It is the generality of the Gaussian distribution that made our study of the one-dimensional random walk so important.